A lattice quantum gravity model with surface-like excitations in 4-dimensional spacetime

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Abstract

A lattice quantum gravity model in 4 dimensional Riemannian spacetime is constructed based on the SU(2) Ashtekar formulation of general relativity. This model can be understood as one of the family of models sometimes called "spin foam models." A version of the action of general relativity in continuum is introduced and its lattice version is defined. A dimensionless "(inverse) coupling" constant is defined so that the value of the action of the model is finite per lattice point. The path integral of the model is expanded in the characters and shown to be written as a sum over surface-like excitations in spacetime. A 3 dimensional version of the model exists and can be reduced to lattice BF theory. The expectation values of some quantities are computed in 3 dimensions and the meanings of the results are discussed. Although the model is studied on a hyper cubic lattice for simplicity, it can be generalized to a randomly triangulated lattice with small modifications.

1 Introduction

A surface theoretic view of non-perturbative quantum gravity was studied some time ago based on 3 dimensional general relativity as a reformulation of the Ponzano-Regge model in Riemannian spacetime [1]. Since 3 dimensional general relativity has no local degrees of freedom, the Ponzano-Regge model is known to be a topological field theory called lattice BF theory in 3 dimensions. A 4 dimensional generalization was made by Ooguri and is known as lattice BF theory in 4 dimensions, also a topological field theory [2]. This Ooguri model also allows the surface theoretic view. However, since 4 dimensional general relativity contains local degrees of freedom, the Ooguri model is not a quantization of general relativity.

In order to pursue 4 dimensional generalizations of the surface theoretic view of the Ponzano-Regge model with local degrees of freedom, some investigations have been being made. They can be divided roughly to four strategies. One [3] is the use of canonical theory called loop quantum gravity [4] based on Ashtekar formulation of general relativity [5]. It uses a Hamiltonian of the canonical theory to compute quantum amplitudes. In this strategy, the definition of "correct" Hamiltonian is still uncertain although the canonical theory is able to provide physical interpretations. Another [6, 7, 8] is the use of the Plebanski action of general relativity in Reimannian spacetime. This action has the form of BF theory with additional constraint

terms [9]. It computes a path integral of the action. In this strategy, a well regulated computation of the constraint terms is still missing and, in addition, the action functional is available only in Reimannian spacetime. Another [10, 11] is the use of "quantized" lattice BF theory and additional "quantum level" constraint conditions. In this strategy, the formulation is strictly mathematical and it necessitates help from other theories for physical interpretations. The other [12] is based on physical reasoning such as causality and statistical criticality. This strategy wants a systematic formulation.

In the present work, a fifth strategy is applied to construct a surface theoretic model for 4 dimensional quantum gravity. We introduce a version of the action of general relativity in continuum. This action is derived from the action proposed some time ago by Samuel and by Jacobson and Smolin [13]. The Ashtekar formulation can be constructed from the latter. We define a lattice version of this action and its path integral. The variables integrated are closely related to the SU(2) (real valued) variables of the Ashtekar formulation. However, the model avoids the use of Hamiltonian of canonical theory, which is a main difficulty so far for quantization. The spacetime the model assumes has signature $(+1, +1, +1, +1)$, Riemannian spacetime. In order to make the lattice action finite per lattice point, a dimensionless "(inverse) coupling" constant is defined in analogy to Wilson's formulation of lattice gauge theory [14]. The finiteness of the action with finite lattice allows the path integral without gauge fixing finite. The path integral of the model is computed as a character expansion and shown to be written as a sum over surface-like excitations in spacetime.

We show that a 3 dimensional version of the model exists and can be reduced to lattice BF theory. In this sense, the model is a 4 dimensional generalization of the Ponzano-Regge model with local degrees of freedom. The expectation values of four quantities are computed in 3 dimensions. Two of them are basic variables of the theory and are SU(2) gauge dependent quantities. Another is the Wilson loop. It is SU(2) gauge independent but diffeomorphism dependent. The other is the action itself. It is invariant under not only SU(2) gauge transformations but also diffeomorphisms. The first three are shown to be always zero unless an appropriate gauge is fixed. The other is zero if the (inverse) coupling constant is brought to an appropriate limit. These are very simple results but important checks for consistency. We discuss the meaning of these results. Note that this kind of computations have not been performed in the other strategies while the computation of the path integral (or sometimes called transition amplitude, projector or partition function) is always focussed on in all the strategies.

The model is defined on a fixed hyper cubic lattice for simplicity. However, the construction can be generalized to a randomly triangulated lattice with small modifications. Further more, the sum over different triangulations can be done, if necessary, by applying a technology developed in a recent work [15].

This paper is organized as follows. In Sec. 2, we restructure the action functional of general relativity so that it can be utilized to define the model. In Sec. 3, the model is defined and its path integral is studied. In Sec. 4, three dimensional version of the model is studied. In Sec. 5, we conclude the work. In Appendix A, mathematical

formulae utilized in this work are listed.

2 The action of general relativity

The action we make use of is the one proposed by Samuel and by Jacobson and Smolin [13]. It is the Palatini action but an additional term. The additional term is a topological term and not responsible to the local degrees of freedom of general relativity. From the Palatini action, one can construct a canonical theory by splitting space and time. However, it contains constraints of the second class in addition to those of the first class in the sense of Dirac. Therefore, the constrained phase space is not well defined unless the second class constraints are explicitly solved.

On the other hand, from the present action, one can construct Ashtekar's canonical theory, which contains only constrains of the first class. It means that the constrained phase space is well defined in the sense that it is invariant under transformations produced by Hamiltonian and diffeomorphism generators. The difference between the two canonical theories is due to the canonical transformation produced by the topological term in the Samuel-Jacobson-Smolin action. We restructure this action for the use in the following section.

Let us introduce variables we need. We use tetrad field e_μ^I rather than metric $g_{\mu\nu}$. They are related with each other by $g_{\mu\nu} = e_\mu^I e_\nu^J \eta_{IJ}$. Here η_{IJ} is the Euclidean or Minkowskian metric of "internal" spacetime depending on the value of σ with signature $(\sigma, +1, +1, +1)$. The capital alphabets $I, J \cdots$ are used for internal spacetime indices $\{0,1,2,3\}$ and the Greek letters are for spacetime indices $\{0,1,2,3\}$. Below, we also use the lower case alphabets $i, j \cdots$ for the space componets of the internal spacetime indices $\{1,2,3\}$ and for the adjoint indices of SU(2) group.

The so-called spin-connection ω_μ^{IJ} is written in terms of the tetrad field as follows.

$$\begin{aligned} \omega_\mu^{IJ} &:= 2e^{\sigma[I}\partial_{[\mu}e_{\sigma]}^{J]} - e^{\sigma I}e^{\nu J}e_{\mu K}\partial_{[\sigma}e_{\nu]}^K \\ &= \sigma\epsilon^{\sigma\alpha\beta\gamma}\epsilon^{IJ}{}_{MN}e_\alpha^M e_\beta^N \left(e_\gamma^K \partial_{[\mu}e_{\sigma]K} + \frac{1}{2}e_\mu^K \partial_\gamma e_{\sigma K}\right). \end{aligned} \tag{1}$$

Here $e^{\mu I}$ is the inverse tetrad. The Riemann tensor $R_{\mu\nu}^{IJ}$ is written in terms of the spin-connection as follows.

$$R_{\mu\nu}^{IJ} := 2\partial_{[\mu}\omega_{\nu]}^{IJ} + 2\omega_{[\mu}^{IK}\omega_{\nu]K}{}^J. \tag{2}$$

In terms of these traditional variables, we define connection and curvature variables A_μ^{IJ} and $F_{\mu\nu}^{IJ}$ as follows.

$$A_\mu^{IJ} := \omega_\mu^{IJ} + \frac{1}{2}\gamma\epsilon^{IJ}{}_{KL}\omega_\mu^{KL} = \frac{1}{2}\gamma\epsilon^{IJ}{}_{KL}A_\mu^{KL} + (1-\gamma^2\sigma)\omega_\mu^{IJ}, \tag{3}$$

$$F_{\mu\nu}^{IJ} := R_{\mu\nu}^{IJ} + \frac{1}{2}\gamma\epsilon^{IJ}{}_{KL}R_{\mu\nu}^{KL} = \frac{1}{2}\gamma\epsilon^{IJ}{}_{KL}F_{\mu\nu}^{KL} + (1-\gamma^2\sigma)R_{\mu\nu}^{IJ}, \tag{4}$$

$$F_{\mu\nu}^{IJ} = 2\partial_{[\mu}A_{\nu]}^{IJ} + A_{[\mu}^{IK}A_{\nu]K}{}^J + (1-\gamma^2\sigma)\omega_{[\mu}^{IK}\omega_{\nu]K}{}^J. \tag{5}$$

Here γ is a parameter and chosen to be $\gamma = 1$ to relate the model with SU(2) (real valued) Ashtekar formulation. If one chooses the value such that $\gamma^2\sigma = 1$, then the variables A_μ^{IJ} and $F_{\mu\nu}^{IJ}$ are self-dual with respect to the internal spacetime indices. We keep γ unspecified in the course of development within this section.

Among the components of A_μ^{IJ} and $F_{\mu\nu}^{IJ}$, we use specifically A_μ^{0i} and $F_{\mu\nu}^{0i}$. $F_{\mu\nu}^{0i}$ is written in terms of A_μ^{0i} and ω_μ^{IJ} as follows.

$$F_{\mu\nu}^{0i} = 2\partial_{[\mu}A_{\nu]}^{0i} - \sigma\gamma\epsilon^{0i}{}_{jk}A_\mu^{0j}A_\nu^{0k} + (1-\gamma^2\sigma)\left[2\omega_{[\mu}^{0k}\omega_{\nu]k}{}^{i} + \frac{1}{2}\gamma\epsilon^{0k}{}_{mn}\omega_{[\mu}^{mn}\omega_{\nu]k}{}^{i}\right]. \tag{6}$$

The pull-back of A_μ^{0i} to foliated space of spacetime is precisely the SU(2) (real valued) connection variables of the Ashtekar canonical formulation.

Let us explicitly write the action. First, the Palatini action derived from the Hilbert-Einstein action is

$$\begin{aligned} S_0[e,\omega] &:= \frac{1}{l_p^2}\int d^4x\sqrt{\sigma g}R = \frac{1}{l_p^2}\int d^4x e e_I^\mu e_J^\nu R_{\mu\nu}^{IJ} \\ &= \frac{1}{2l_p^2}\int d^4x \epsilon^{\mu\nu\lambda\sigma}\epsilon_{IJKL}e_\mu^I e_\nu^J R_{\lambda\sigma}^{KL}. \end{aligned} \tag{7}$$

Here, g is the determinant of metric $g_{\mu\nu}$, R is the scalar curvature, l_p is the Planck length constant, e is the determinant of e_μ^I and $\epsilon^{\mu\nu\lambda\sigma}$ is the alternating density. By adding a topological term as mentioned above, the action we make use of is

$$\begin{aligned} S[e,A] &:= \frac{1}{\gamma l_p^2}\int d^4x\epsilon^{\mu\nu\lambda\sigma}e_{\mu I}e_{\nu J}\left[\frac{1}{2}\gamma\epsilon^{IJ}{}_{KL}R_{\lambda\sigma}^{KL} + R_{\lambda\sigma}^{IJ}\right] \\ &= \frac{1}{\gamma l_p^2}\int d^4x\epsilon^{\mu\nu\lambda\sigma}e_{\mu I}e_{\nu J}F_{\lambda\sigma}^{IJ}. \end{aligned} \tag{8}$$

This is the Samuel-Jacobson-Smolin action. By dividing space and time indices for internal spacetime and writing in terms of e_μ^0, e_μ^i and A_μ^{0i}, the action becomes

$$\begin{aligned} S[e,A] &= \frac{1}{\gamma l_p^2}\int d^4x\epsilon^{\mu\nu\lambda\sigma}\left[2e_{\mu0}e_{\nu i}F_{\lambda\sigma}^{0i} + e_{\mu i}e_{\nu j}F_{\lambda\sigma}^{ij}\right] \\ &= \frac{1}{\gamma l_p^2}\int d^4x\epsilon^{\mu\nu\lambda\sigma}\left[2e_{\mu0}e_{\nu i}F_{\lambda\sigma}^{0i} + e_{\mu i}e_{\nu j}\left[\gamma\epsilon^{ij}{}_{0k}F_{\lambda\sigma}^{0k} + (1-\gamma^2\sigma)R_{\lambda\sigma}^{ij}\right]\right] \\ &= \frac{1}{\gamma l_p^2}\int d^4x\epsilon^{\mu\nu\lambda\sigma}\left[\sigma\left(2e_{[\mu}^0e_{\nu]i} + \gamma\epsilon^0{}_{ijk}e_\mu^je_\nu^k\right)F_{\lambda\sigma}^{0i} + (1-\gamma^2\sigma)e_{\mu i}e_{\nu j}R_{\lambda\sigma}^{ij}\right] \\ &= \frac{1}{\gamma l_p^2}\int d^4x\epsilon^{\mu\nu\lambda\sigma}\left\{\sigma\left(2e_{[\mu}^0e_{\nu]i} + \gamma\epsilon^0{}_{ijk}e_\mu^je_\nu^k\right)\left(2\partial_{[\lambda}A_{\sigma]}^{0i} - \sigma\gamma\epsilon^{0i}{}_{jk}A_\lambda^{0j}A_\sigma^{0k}\right)\right. \\ &+(1-\gamma^2\sigma)\left[\sigma\left(2e_{[\mu}^0e_{\nu]i} + \gamma\epsilon^0{}_{ijk}e_\mu^je_\nu^k\right)\left(2\omega_{[\lambda}^{0k}\omega_{\sigma]k}{}^{i} + \frac{1}{2}\gamma\epsilon^{0k}{}_{mn}\omega_{[\lambda}^{mn}\omega_{\sigma]k}{}^{i}\right)\right. \\ &\left.\left.+e_{\mu i}e_{\nu j}R_{\lambda\sigma}^{ij}\right]\right\}. \end{aligned} \tag{9}$$

Here, we consider ω_μ^{IJ} and $R_{\mu\nu}^{ij}$ as functions of e_μ^0 and e_μ^i. Note that if $\gamma^2\sigma = 1$, then the terms proportional to $(1-\gamma^2\sigma)$ go away and the action gets simplified. We will utilize the case that $\gamma = 1$ and $\sigma = 1$, Riemannian spacetime.

3 The model

3.1 The action

The action discussed in the previous section for Riemannian spacetime ($\sigma = +1$) with $\gamma = 1$ is

$$S_R[e, A] = \frac{1}{l_p^2} \int d^4x \epsilon^{\mu\nu\lambda\sigma} \left(2e^0_{[\mu} e_{\nu]i} + \epsilon^0{}_{ijk} e^j_\mu e^k_\nu \right) \left(2\partial_{[\lambda} A^i_{\sigma]} - \epsilon^{0i}{}_{jk} A^j_\lambda A^k_\sigma \right). \quad (10)$$

Hereafter, we write A^i_μ for A^{0i}_μ. We note that if $2e^0_{[\mu} e^i_{\nu]} + \epsilon^{0i}{}_{jk} e^j_\mu e^k_\nu$ is replaced by a 2-form variable denoted by $B^i_{\mu\nu}$ with the condition that $\epsilon^{\mu\nu\lambda\sigma} B^i_{\mu\nu} B^j_{\lambda\sigma}$ is positionwisely proportional to δ^{ij}, then the resulting action is precisely the Plebanski action. This additional condition is supposed to restore $B^i_{\mu\nu} = 2e^0_{[\mu} e^i_{\nu]} + \epsilon^{0i}{}_{jk} e^j_\mu e^k_\nu$. Ways of constructing a surface theoretic model starting from the Plebanski action have been explored. The condition is usually imposed with Lagrange multipliers. However, it has turned out that the inclusion of the condition in terms of Lagrange multilpiers is still a difficult task. In the present work, we do not follow any of those ways. Instead, we define a corresponding action directly on the lattice without introducing $B^i_{\mu\nu}$ and utilizing Lagrange multipliers.

In order to define the model, we rewrite the action in a slightly compact fashion so that the construction of the lattice version is straightforward. An idea is the following. The tetrad variables (or their modifications) are often considered su(2) algebra valued. For example, in the Ashtekar formulation, the pull-back of the (inverse) tetrads to space have su(2) algebra indices and , in the Plebanski formulation, the 2-form variable $B^i_{\mu\nu}$ are su(2) algebra valued. However, we do not consider here the tetrad variables su(2) algebra valued but do consider them SU(2) group valued. If one defines $e_\mu := e^0_\mu + i\sigma_i e^i_\mu$ with the Pauli matrices σ_i ($i = 1, 2, 3$), then the complicated expression $2e^0_{[\mu} e^i_{\nu]} + \epsilon^{0i}{}_{jk} e^j_\mu e^k_\nu$ becomes just the non-trace part of $e^\dagger_\mu e_\nu$. If e_μ is normalized so that $e^0_\mu e^0_\mu + e^1_\mu e^1_\mu + e^2_\mu e^2_\mu + e^3_\mu e^3_\mu = 1$, then it is an SU(2) group element. This small change of view dramatically simplifies the construction of the model and makes the extension to other dimensional spacetimes straightforward.

Before introducing the lattice, let us count the number of degrees of freedom of the variables. e^I_μ and A^i_μ have respectively 16 and 12 degrees of freedom per spacetime point. From thses variables, if one constructs canonical pairs of variables, one finds 2 physical degrees of freedom from e^I_μ and 2 from A^i_μ. The other degrees of freedoms are for the constraints and gauge trajectries in addition to the freedom of specifying a spacetime foliation. Let us be slightly more concrete. Out of 16 degrees of freedom of e^I_μ, 3 fix a spacetime foliation and 4 are used for Lagrange multipliers imposing three diffeomorphism and a Hamiltonian constraints. Out of 12 degrees of freedom of A^i_μ, 3 are used for Lagrange multipliers imposing three Gauss gauge constraints. Hence, 9 degrees of freedom in each are left. These are exactly the number of degrees of freedom of the Ashtekar canonical theory. Then by subtracting 7 for the constraints and 7 for the gauge degrees of freedom, one finds the correct number of physical degrees of freedom.

3.2 Lattice

We fix a lattice. We do not take sum over different lattices. In general, the model can be defined on a randomly triangulated lattice. However, technically it is difficult, if not impossible, to compute the quantum amplitudes analytically or even numerically if the structure of lattice varies from place to place. For this reason, we use a hyper cubic lattice. First we introduce a pair of hyper cubic lattices. One is dual to the other. Denote them Δ and Δ^* respectively. Then we further restrict ourselves to the case that Δ^* is Δ itself. This is possible because the dual lattice of a hyper cubic lattice is a hyper cubic lattice. This restriction dramatically simplifies the construction of the model and helps computations in practice.

The lattices Δ and Δ^* are a pair of 4-dimensional hyper cubic lattices dual to each other. They consist of 0-cells (vertices), 1-cells (edges), 2-cells (square faces), 3-cells (cubes) and 4-cells (hyper cubes). Every k-cell of one lattice intersects with a (4-k)-cell of the other lattice at a point inside the cells. Every cell of the lattices has one and only one intersection. The correspondence of k-cells of Δ and (4-k)-cells of Δ^* is one-to-one. The corresponding k-cell of Δ and (4-k)-cell of Δ^* are called dual to each other. There is a limit at which every intersection coincides with one of the vertices so that the two lattices are identical $\Delta = \Delta^*$.

Let us be more specific. Fix Δ and Δ^*. Both have a lattice spacing ε. One is the displacement of the other by the distance $\varepsilon/2$ in all the four directions. Let x denote (the position of) a vertex of Δ. The (positions of) adjacent vertices are $x \pm \varepsilon^\mu$ with $\mu = 0, 1, 2, 3$, where ε^μ is the increment in the μ-th direction with the magnitude ε. Note that $x \pm \varepsilon^0$, for example, is a short hand notation for $(x^0 \pm \varepsilon, x^1, x^2, x^3)$. Let x^* denote (the position) of a vertex of Δ^* at $x + \varepsilon^0/2 + \varepsilon^1/2 + \varepsilon^2/2 + \varepsilon^3/2$. The face of Δ specified by (the positions of) four vertices x, $x + \varepsilon^\mu$, $x + \varepsilon^\nu$ and $x + \varepsilon^\mu + \varepsilon^\nu$ is dual to the face of Δ^* specified by (the positions of) four vertices x^*, $x^* - \varepsilon^\lambda$, $x^* - \varepsilon^\sigma$ and $x^* - \varepsilon^\lambda - \varepsilon^\sigma$, where the indices are chosen such that $\epsilon^{\mu\nu\lambda\sigma} = 1$. There are six pairs of faces for given x up to the orientation of the face. These dual faces are important in our construction of the model. After taking the limit $\Delta^* \to \Delta$ such that x^* coincides with x, the dual faces are still clearly defined on the single lattice Δ. The intersection of a dual pair of faces, one basing at x and the other at x^*, coincides with $x = x^*$ at the limit. Notice that the face basing at x^* and in the $\mu\nu$ plane does not coincide with the face basing at x and in the same plane at the limit. One is in the μ and ν directions while the other is in the $-\mu$ and $-\nu$ directions from the coincident base point $x = x^*$. We will define our model on the lattice $\Delta(= \Delta^*)$.

3.3 The lattice action

We define an action on the lattice Δ so that it converges to the action in continuum as the lattice spacing goes to zero. First, let us define variables on the lattice as follows. For an edge basing at x and pointing at the μ-th direction, define

$$\zeta_\mu(x) := \zeta^0_\mu(x) + i\sigma_i \zeta^i_\mu(x) := \exp[i\varepsilon A^i_\mu(x)\sigma_i/2], \tag{11}$$

$$\eta_\mu(x) := \eta^0_\mu(x) + i\sigma_i\eta^i_\mu(x) := (\beta^{1/2}l_p)^{-1}\varepsilon[e^0_\mu(x) + ie^i_\mu(x)\sigma_i]/\rho_\mu(x), \tag{12}$$

$$\rho_\mu(x) := (\beta^{1/2}l_p)^{-1}\varepsilon\sqrt{|e^0_\mu(x)|^2 + |e^1_\mu(x)|^2 + |e^2_\mu(x)|^2 + |e^3_\mu(x)|^2}. \tag{13}$$

Here, ε is the lattice spacing and σ_i's are the Pauli matrices, that is,

$$\sigma_1 := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 := \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{14}$$

Note that the inside the square root in ρ_μ is equal to the metric element $g_{\mu\mu}$. β is the upper bound of $l_p^{-2}\varepsilon^2 g_{\mu\mu}(\mu = 0, 1, 2, 3)$ and introduced to regularize the path integral defined below. ρ_μ is non-negative and less than 1, and ζ_μ and η_μ are SU(2) matrices normalized such that $|\zeta^0_\mu|^2+|\zeta^1_\mu|^2+|\zeta^2_\mu|^2+|\zeta^3_\mu|^2 = 1$ and $|\eta^0_\mu|^2+|\eta^1_\mu|^2+|\eta^2_\mu|^2+|\eta^3_\mu|^2 = 1$. For a face basing at x and enclosed by four edges in the μ and ν-th direstions, define

$$U_{\mu\nu}(x) := \zeta_\mu(x)\zeta_\nu(x + \varepsilon^\mu)\zeta^\dagger_\mu(x + \varepsilon^\nu)\zeta^\dagger_\nu(x), \tag{15}$$

without summing over μ and ν. Here, ε^μ is the increment in the μ-th direction with the magnitude ε.

The action on the lattice Δ is then defined as follows.

$$S_\Delta[\zeta, \eta, \rho] := -\beta \sum_{x\in\Delta} \epsilon^{\mu\nu\lambda\sigma}\rho_\mu(x)\rho_\nu(x)\mathrm{Tr}[\eta^\dagger_\mu(x)\eta_\nu(x)U_{\lambda\sigma}(x)]. \tag{16}$$

Here, the sums over μ, ν, λ and σ have been performed. This lattice action is finite on the lattice Δ with ε and β fixed if the lattice size is finite. This is because ζ_μ and η_μ are compact SU(2) variables and ρ_μ has lower and upper bounds in addition to the fact that the number of degrees of freedom is finite on the lattice if the lattice size is finite. The finiteness of the action lets the path integral without gauge fixing non-divergent. In other words, the path integral along a gauge trajectory produces just a finite overall multiplicative constant. We call β "inverse coupling" in analogy to Wilson's formulation of lattice gauge theory.

Let us examine if the lattice action really converges to the continuum action as the lattice spacing ε goes to zero. Take the limit $\varepsilon \to 0$. In the limiting process, it can be shown that

$$\beta\rho_\mu(x)\rho_\nu(x)\eta^\dagger_{[\mu}(x)\eta_{\nu]}(x) = l_p^{-2}\varepsilon^2 i\sigma_i\left(2e^0_{[\mu}(x)e^i_{\nu]}(x) + \epsilon^{0i}{}_{jk}e^j_\mu(x)e^k_\nu(x)\right), \tag{17}$$

$$U_{[\lambda\sigma]}(x) \to \frac{1}{2}\varepsilon^2 i\sigma_i\left(2\partial_{[\lambda}A^i_{\sigma]}(x) - \epsilon^{0i}{}_{jk}A^j_\lambda(x)A^k_\sigma(x)\right) + \mathcal{O}(\varepsilon^3). \tag{18}$$

Therefore, the lattice action converges to

$$\begin{aligned} S_\Delta[\zeta, \eta, \rho] \to \ & l_p^{-2}\sum_{x\in\Delta}\varepsilon^4\epsilon^{\mu\nu\lambda\sigma}\left(2e^0_\mu(x)e_{\nu i}(x) + \epsilon^0{}_{ijk}e^j_\mu(x)e^k_\nu(x)\right)\times \\ & \left(2\partial_\lambda A^i_\sigma(x) - \epsilon^{0i}{}_{jk}A^j_\lambda(x)A^k_\sigma(x)\right) + \mathcal{O}(\varepsilon^5). \end{aligned} \tag{19}$$

3.4 The lattice path integral

The exponential of the action can be considered as (an extension of) the graph-cylindrical function discussed in [16]. The integral measure we use for the variable A^i_μ is the Ashtekar-Lewandowski measure [16]. The measure for SU(2)-compactified part of e^0_μ and e^i_μ is analogously defined and that for the other part of them is the one discussed in [3]. The path integral has the form of

$$\int d\mu(A)d\mu(e)\Psi_{\Delta,\psi}(A,e) := \int \prod_{x\in\Delta}\prod_{\mu} d\zeta_\mu(x)d\eta_\mu(x)d\rho^4_\mu(x) \times$$
$$\psi(\zeta(x_1,A),\cdots\zeta(x_l,A);\eta(x_1,e),\cdots\eta(x_m,e);\rho(x_1,e),\cdots\rho(x_n,e)), \qquad (20)$$

where $\Psi_{\Delta,\psi}$ is a cylindrical function defined on the lattice Δ in terms of a complex valued integrable function ψ on $[SU(2)]^l \times [SU(2)]^m \times [0,1]^n$. $d\zeta_\mu$ and $d\eta_\mu$ are the Haar measures for SU(2) elements ζ_μ and η_μ respectively. $d\rho^4_\mu = 4d\rho_\mu\rho^3_\mu$ is the radial integral of the 4-dimensional unit sphere. ρ_μ is integrated from 0 to 1.

The path integral we will compute in the following section is defined as follows.

$$\begin{aligned} Z_\Delta &:= \int d\zeta d\eta d\rho^4 e^{-iS_\Delta[\zeta,\eta,\rho]} \\ &= \int d\zeta d\eta d\rho^4 \prod_{x\in\Delta} e^{i\beta\epsilon^{\mu\nu\lambda\sigma}\rho_\mu(x)\rho_\nu(x)\mathrm{Tr}[\eta^\dagger_\mu(x)\eta_\nu(x)U_{\lambda\sigma}(x)]}. \end{aligned} \qquad (21)$$

Note that we integrate the exponential oscillation form e^{iS} rather than the exponential decay form e^{-S}. The latter is commonly and successfully used for Euclidean quantum field theories whose action has quadratic form. However, the action of general relativity is not quadratic but linear in each of the variables. Because of this fact, the action is not bound from the below and the use of the exponential decay form runs into serious technical problems. In addition, it unlikely contains the classical limit since the value of the action for classical solutions is zero. The former is used in other strategies in quantum gravity [1, 2, 3, 6, 7, 8, 10, 11, 15] even for Riemannian spacetime. In the exponential oscillation form, large magnitudes of the action less contribute to the path integral. In order to examine possible consequences of the use of the exponential decay form, we simply replace β by $i\beta$ in the results of the use of the exponential oscillation form.

In terms of the path integral, define the expectation value of an observable X as follows.

$$\langle X\rangle_\Delta := Z^{-1}_\Delta \int d\zeta d\eta d\rho^4 X[\zeta,\eta,\rho]e^{-iS_\Delta[\zeta,\eta,\rho]}. \qquad (22)$$

In particular, we are interested in four quantities: $\langle\eta_\mu\rangle$, $\langle\zeta_\mu\rangle$, $\langle\mathrm{Tr}U_{\mu\nu}\rangle$ and $\langle S_\Delta\rangle$. The first two are basic variables of the theory and are SU(2) dependent quantities. They should vanish. This is because there is a symmetric structure along the SU(2) gauge trajectories. It is known that the expactation value of the basic variables of lattice QCD vanishes unless an appropriate gauge is fixed. This fact is often utilized to study gauge fixing ambiguities on lattice. The same thing should happen here since

the Gauss gauge constraint structure of the present theory is common with the SU(2) version of QCD.

The third is (a smallest of) the Wilson loop. It is SU(2) gauge independent and a good physical observable for Gauss gauge invariant theories such as lattice QCD. Here, it is possible that the expectation value of the Wilson loop vanishs unless an appropriate gauge is fixed. This would mean that the Gauss gauge invariance is not enough to be a physical observable for quantum gravity and a possible symmetric structure along diffeomorphism gauge trajectories annihilates the expectation value of the Wilson loop. Quantum gravity is invariant not only under Gauss gauge transformations but also under spacetime diffeomorphisms. The Wilson loops are not diffeomorphism invariant and hence cannot be physical observables of quantum gravity.

The last one is the action of the theory. It is invariant under SU(2) gauge transformations and diffeomorphisms and the only known local physical observable in general relativity. It should vanish if the inverse coupling constant is brought to infinity but diverge in the same limit if the path integral has the exponential decay form. The former would be an indication that the path integral successfully eliminates the degrees of freedom constrained since the value of the action on the constraint surface must be identically zero. The latter would be an indication that the path integral of exponential decay form cannot capture the classical solutions. These are non-trivial checks for consistency. These facts are examined in 3 dimensions.

3.5 Character expansion

We compute the path integral as a character expansion and show that it can be written as a sum over surface-like excitations in spacetime. First we rewrite the path integral as follows.

$$\begin{aligned} Z_\Delta &:= \int d\zeta d\eta d\rho^4 e^{-iS_\Delta[\zeta,\eta,\rho]} \\ &= \int d\zeta d\eta d\rho^4 \prod_{x\in\Delta}\prod_{(\mu\nu\lambda\sigma)} e^{i2\beta\rho_\mu(x)\rho_\nu(x)\mathrm{Tr}[\eta^\dagger_\mu(x)\eta_\nu(x)U_{\lambda\sigma}(x)]} e^{-i2\beta\rho_\mu(x)\rho_\nu(x)\mathrm{Tr}[\eta^\dagger_\nu(x)\eta_\mu(x)U_{\lambda\sigma}(x)]}. \end{aligned} \tag{23}$$

Here, $(\mu\nu\lambda\sigma)$ denotes the even permutations of (0123); there are six terms, that is, (0123), (0231), (0312), (1203), (3102), and (2301). Then use the formula (36) to expand to the characters as follows.

$$Z_\Delta = \int d\zeta d\eta d\rho^4 \prod_{x\in\Delta}\prod_{(\mu\nu\lambda\sigma)}\sum_{l_{\mu\nu},l_{\nu\mu}} \Gamma_{\mu\nu}(\beta;\{l,\rho\})\chi_{l_{\mu\nu}}(\eta^\dagger_\mu\eta_\nu U_{\lambda\sigma})\chi_{l_{\nu\mu}}(\eta^\dagger_\nu\eta_\mu U_{\lambda\sigma}), \tag{24}$$

with

$$\Gamma_{\mu\nu}(\beta;\{l,\rho\}) := \frac{(2l_{\mu\nu}+1)(2l_{\nu\mu}+1)(-1)^{l_{\mu\nu}-l_{\nu\mu}}}{4\beta^2\rho_\mu^2\rho_\nu^2} J_{2l_{\mu\nu}+1}(4\beta\rho_\mu\rho_\nu)J_{2l_{\nu\mu}+1}(4\beta\rho_\mu\rho_\nu). \tag{25}$$

Here, $\chi_j(U)$ is the character of SU(2) element U in the spin-j representation and $l_{\mu\nu}$ and $l_{\nu\mu}$ are spins taking values 0, $\frac{1}{2}$, 1, $\frac{3}{2}\cdots$. $J_m(x)$ is the Bessel function of the first kind. One of its definitions is given by (37).

From (24), one can understand how surface-like excitations emerge. $U_{\lambda\sigma}(x)$ is a parallel transport along the four edges enclosing a face of the lattice. The face is in the $\lambda\sigma$-plane and bases at x. Let $f_{\lambda\sigma}(x)$ denote this face. (We do not distinguish the orientations; $f_{\lambda\sigma}(x) \equiv f_{\sigma\lambda}(x)$.) Associate the spin $l_{\mu\nu}$ of $\chi_{l_{\mu\nu}}(\eta^\dagger_\mu(x)\eta_\nu(x)U_{\lambda\sigma}(x))$ to the face $f_{\lambda\sigma}(x)$. In the same way, $l_{\nu\mu}$ is also associated to the same face. By integrating η_μ at x, one introduces an intertwiner to combine six spins $l_{\mu\nu}$ and $l_{\nu\mu}$ ($\nu \neq \mu$ for given μ) associated with $f_{\lambda\sigma}(x)$ (λ and σ are such that $\epsilon^{\mu\nu\lambda\sigma} = 1$). By integrating ζ_μ shared by six faces $f_{\mu\nu}(x)$ and $f_{\mu\nu}(x - \varepsilon^\nu)$ ($\nu \neq \mu$ for given μ), one introduces an intertwiner to combine the twelve spins associated with the six faces. Therefore, after integrating all η's and ζ's, the spins and the intertwiners combining the spins are left. Two spins are associated with a single face and two intertwiners are associated with a single edge. These facts are more clearly understood in the 3 dimensional version of the model as examined in Sec. 4. The path integral is now written as a sum over spins and intertwiners associated with faces and edges respectively. A set of faces jointed by edges represent a surface-like object, mathematically called 2-dimensional piece-wise linear cell complex [11]. A surface-like excitation is a 2-dimensional piece-wise linear cell complex with faces labeled by non-zero spins and with edges labeled by intertwiners. A face with spin-0 means the absence of the face in a surface-like excitation.

The integrations of η's and ζ's can be explicitly done by using the formulae (38), (39) and (40). The resulting coefficiens contain the spins associated with faces and other spins representing intertwiners associated with edges. The coefficients are expressed in terms of the so-called 3j-coefficients (41). The integrations of ρ's are difficult because of the presence of two ρ's in the argument of the Bessel functions. This difficulty is absent in the 3 dimensional version of the model.

4 The model in 3-dimensions

4.1 The action and path integral

From the action in 4 dimensions, it is straightforward to write its 3 dimensional version. We simply replace $l_p^{-2}d^4x\epsilon^{\sigma\lambda\mu\nu}e^\dagger_\sigma e_\lambda$ in the 4 dimensional action by $l_p^{-1}d^3x\epsilon^{\lambda\mu\nu}e_\lambda$. We compute the corresponding path integral on the lattice and show that it is reduced to lattice BF theory in 3 dimensions in order to claim that 3 dimensional version of the model exists and is a correct quantum gravity model.

The action of general relativity in 3 dimensional Riemannian spacetime is defined as follows.

$$S_{R_3} := \frac{1}{l_p}\int d^3x\epsilon^{\lambda\mu\nu}e_{\lambda i}\left(2\partial_{[\mu}A^i_{\nu]} - \epsilon^i{}_{jk}A^j_\mu A^k_\nu\right). \qquad (26)$$

Here, the Greek letter indices are $\{0,1,2\}$ for spacetime and lower alphabet indices are $\{1,2,3\}$ for internal spacetime.

Let us count the number of degrees of freedom. Out of 9 degrees of freedom of e^i_μ at each spacetime point, 3 are for Lagrange multipliers imposing two diffeomorphism and a Hamiltonian constraints. Out of 9 degrees of freedom of A^i_μ at each spacetime point, 3 are for Lagrange multipliers imposing 3 Gauss constraints. Hence, 6 degrees of freedom in each are left. They are cancelled by the 6 constraints and 6 gauge degrees of freedom and one finds no local degrees of freedom consistently with general relativity. Note that in canonical theory one sees the vectors specifying a spacetime foliation multiplicative e^i_μ. Therefore, the integral over all foliations is taken into account by the integrals of e^i_μ.

The lattice action is defined as follows.

$$S_{\Delta_3}[\zeta,\eta,\rho] := -\beta \sum_{x\in\Delta_3} \epsilon^{\lambda\mu\nu}\rho_\lambda(x)\mathrm{Tr}[\eta_\lambda(x)U_{\mu\nu}(x)]. \tag{27}$$

Here, Δ_3 denotes the 3 dimensional cubic lattice. Note that the definitions of the variables are the same as for the 4 dimensional model, (11),(12) and (13), and the additional degrees of freedom η^0_μ cancel in the lattice action and do not play any role. Then the lattice path integral is defined as follows.

$$\begin{aligned} Z_{\Delta_3} &:= \int d\zeta d\eta d\rho^4 \prod_{x\in\Delta_3} e^{i\beta\epsilon^{\lambda\mu\nu}\rho_\lambda(x)\mathrm{Tr}[\eta_\lambda(x)U_{\mu\nu}(x)]} \\ &= \int d\zeta d\eta d\rho^4 \prod_{x\in\Delta_3} \prod_{\substack{(\lambda\mu\nu)=(012),\\(120),(201)}} e^{i\beta\rho_\lambda\mathrm{Tr}[\eta_\lambda U_{\mu\nu}]}e^{-i\beta\rho_\lambda\mathrm{Tr}[\eta_\lambda U_{\nu\mu}]}. \end{aligned} \tag{28}$$

Then use the formula (36) to expand to the characters as follows.

$$Z_{\Delta_3} = \int d\zeta d\eta d\rho^4 \prod_{x\in\Delta_3} \prod_{\substack{(\lambda\mu\nu)=(012),\\(120),(201)}} \sum_{l_{\mu\nu},l_{\nu\mu}} \Gamma_\lambda(\beta,\{l,\rho\})\chi_{l_{\mu\nu}}(\eta_\lambda U_{\mu\nu})\chi_{l_{\nu\mu}}(\eta^\dagger_\lambda U_{\mu\nu}), \tag{29}$$

with

$$\Gamma_\lambda(\beta,\{l,\rho\}) := \frac{(2l_{\mu\nu}+1)(2l_{\nu\mu}+1)(-1)^{l_{\mu\nu}-l_{\nu\mu}}}{\beta^2\rho^2_\lambda} J_{2l_{\mu\nu}+1}(2\beta\rho_\lambda)J_{2l_{\nu\mu}+1}(2\beta\rho_\lambda). \tag{30}$$

Here, one can understand the emergence of surface-like excitations, in other words, 2 dimensional piece-wise linear cell complexs with faces labeled by non-zero spins and with edges labeled by intertwiners. $U_{\mu\nu}$ is a parallel transport along the four edges enclosing a face of the lattice. The face is in the $\mu\nu$-plane and bases at x. Let $f_{\mu\nu}(x)$ denote this face. Associate the spin $l_{\mu\nu}$ of $\chi_{l_{\mu\nu}}(\eta_\lambda U_{\mu\nu})$ to the face $f_{\mu\nu}(x)$. In the same way, $l_{\nu\mu}$ is also associated to the same face. By integrating η_λ at x, one introduces an intertwiner to combine two spins $l_{\mu\nu}$ and $l_{\nu\mu}$ associated to $f_{\mu\nu}(x)$ (μ and ν are such that $\epsilon^{\lambda\mu\nu}=1$). By integrating ζ_μ shared by four faces $f_{\mu\nu}(x)$ and

$f_{\mu\nu}(x-\varepsilon^\nu)$ ($\nu \neq \mu$ for given μ), one introduces an intertwiner to combine the eight spins associated with the four faces. Therefore, after integrating all η's and ζ's, the spins and the intertwiners combining the spins are left. The path integral is now written as a sum over spins and intertwiners representing surface-like excitations.

The integrations of η's, ζ's and ρ's can be explicitly done by using the formulae (38), (39) and (40). Integrate η's and ρ's as follows.

$$Z_{\Delta_3} = \int d\zeta \prod_{x\in\Delta_3} \prod_{\substack{(\mu\nu)=(12),\\(20),(01)}} \sum_{l_{\mu\nu}} \Delta_{\mu\nu}(\beta)(2l_{\mu\nu}+1)\chi_{l_{\mu\nu}}(U_{\mu\nu}U_{\mu\nu}), \tag{31}$$

with

$$\Delta_{\mu\nu}(\beta) := \int_0^1 d\rho_\lambda^4 \left[\frac{J_{2l_{\mu\nu}+1}(2\beta\rho_\lambda)}{\beta\rho_\lambda}\right]^2 = \beta^{-4}\int_0^{2\beta} dkkJ_{2l_{\mu\nu}+1}(k)J_{2l_{\mu\nu}+1}(k)$$
$$\to \beta^{-4}\delta(0) \ \text{ as } \ \beta\to\infty. \tag{32}$$

In the limit $\beta\to\infty$, $\Delta_{\mu\nu}$ loses the dependence on $l_{\mu\nu}$ and the path integral can be rewritten as follows.

$$\lim_{\beta\to\infty} Z_{\Delta_3} = \int d\zeta \prod_{x\in\Delta_3} \prod_{\substack{(\mu\nu)=(12),\\(20),(01)}} \sum_{l_{\mu\nu}} (2l_{\mu\nu}+1)[\chi_{l_{\mu\nu}}(U_{\mu\nu}) + \chi_{l_{\mu\nu}}(-U_{\mu\nu})], \tag{33}$$

up to an overall multiplicative constant. The first half is well known form for 3-dimensional lattice BF theory and is meant that spacetime is flat $U_{\mu\nu}=1$. In addition, we have another term also meaning that spacetime is flat but $U_{\mu\nu}=-1$. The presence of the additional term is due to the definition of the lattice action. Because of the definition of the lattice action, the path integral contains a form analogous to $\int dxe^{ix\sin\theta} = \delta(\theta)+\delta(\theta-\pi)$ instead of the well known form analogous to $\int dxe^{ix\theta} = \delta(\theta)$. We have understood that the path integral of the model in 3 dimensions can be reduced to (a generalization of) that of 3 dimensional lattice BF theory.

4.2 Expectation values

We compute the expectation values of four quantities, $\langle\eta_\mu\rangle$, $\langle\zeta_\mu\rangle$, $\langle\mathrm{Tr}U_{\mu\nu}\rangle$ and $\langle S_{\Delta_3}\rangle$, and find all of them vanish. However, their meanings are different as discussed below. Remember that the path integral contains the integrations of 18 degrees of freedom per lattice point (9 for ζ_μ and 9 for η_μ and ρ_μ). If one translates them into the canonical theory, 3 of ζ_μ and 3 of η_μ and ρ_μ are Lagrange multipliers imposing the 6 constraints per lattice point. The other degrees of freedom are 6 degrees of freedom constrained and 6 on the constraint surface. The degrees of freedom constrained do not contribute to the path integral. When we compute the expectation values, the integrations over the degrees of freedom on the constraint surface make non trivial

contributions. All of the local degrees of freedom on the constraint surface are gauge degrees of freedom in the 3 dimensional case.

$\langle\eta_\mu\rangle$ and $\langle\zeta_\mu\rangle$ are basic variables of the theory. They are SU(2) gauge dependent quantities. It is known that the expectation value of basic variable of lattice QCD vanishes unless an appropriate gauge is fixed because there is a symmetric structure along the SU(2) gauge trajectories. The same Gauss gauge constraint structure is also present in general relativity in the present formulation. Therefore, $\langle\eta_\mu\rangle$ and $\langle\zeta_\mu\rangle$ should vanish if the model is consistent.

Let us compute the expectation values of $\langle\eta_\mu\rangle$ and $\langle\zeta_\mu\rangle$. Insert $\eta_0(x)$ into (29) and integrate $\eta_0(x)$. This results in $l_{21} = l_{12} \pm \frac{1}{2}$ associated with $f_{12}(x)$. On the other hand, the integrations of $\eta_2(x)$, $\eta_0(x-\varepsilon^2)$ and $\eta_2(x-\varepsilon^0)$ result in $l_{10} = l_{01}$ associated with $f_{01}(x)$, $l_{21} = l_{12}$ associated with $f_{12}(x-\varepsilon^2)$ and $l_{10} = l_{01}$ associated with $f_{01}(x-\varepsilon^0)$ respectively. $f_{12}(x)$, $f_{01}(x)$, $f_{12}(x-\varepsilon^2)$ and $f_{01}(x-\varepsilon^0)$ share the edge where $\zeta_1(x)$ is defined. Hence, the integration of $\zeta_1(x)$ introduces an intertwiner combining the spins associated with the four faces. However, no intertwiner consistent with these spins exists because the sum of the 8 spins is a half integer. Therefore, we conclude $\langle\eta_0\rangle = 0$. In the same way, we find $\langle\eta_1\rangle = \langle\eta_2\rangle = 0$.

In order to compute $\langle\zeta_0\rangle$, insert $\zeta_0(x)$ into (31) and integrate $\zeta_0(x)$. The edge where $\zeta_0(x)$ is defined is shared by the four faces $f_{01}(x)$, $f_{02}(x)$, $f_{01}(x-\varepsilon^1)$ and $f_{02}(x-\varepsilon^2)$. Since each of the four faces is associated with two equal spins after the integrations of η's, the sum of the 8 spins associated with the four faces is an integer. There is no intertwiner combining an integer and $\frac{1}{2}$ to produce the spin-0. Therefore, we conclude $\langle\zeta_0\rangle = 0$. In the same way, we find $\langle\zeta_1\rangle = \langle\zeta_2\rangle = 0$.

$\langle\mathrm{Tr}U_{\mu\nu}\rangle$ is the expectation value of the Wilson loop and is an SU(2) gauge independent quantity. It is a good physical observable for Gauss gauge invariant theories such as lattice QCD. However, the Gauss gauge invariance is not enough to be a physical observable for quantum gravity. Physical observables in quantum gravity must be invariant not only under SU(2) gauge transformations but also under spacetime diffeomorphisms. The Wilson loop is not invariant under diffeomorphisms. Therefore, it is possible that $\langle\mathrm{Tr}U_{\mu\nu}\rangle$ vanishes because of a symmetric structure along diffeomorphism gauge trajectories unless an appropriate gauge is fixed. If it is the case, the vanishing result can be understood as a consequence of the fact that quantum gravity is a spacetime diffeomorphism invariant theory.

The reasoning for $\langle\zeta_\mu\rangle$ to vanish also holds for $\langle\mathrm{Tr}U_{\mu\nu}\rangle$ to vanish. Since every face is associated with two equal spins after the integrations of η's, there is no way of combining the 8 spins associated with the four faces and spin $\frac{1}{2}$ of $\mathrm{Tr}U_{\mu\nu}$ to produce the spin-0. Therefore, we find $\langle\mathrm{Tr}U_{\mu\nu}\rangle = 0$. The fact that every face is associated with two spins is a notable difference of the model from lattice QCD. One of the reasons for this fact is the presence of the variable e^i_μ in addition to the connection variable A^i_μ in the action. Note that $\langle\mathrm{Tr}U_{\mu\nu}\rangle = 0$ does not contradict to the fact that the same quantity evaluated with flat connection is not zero but one since this loop can be contracted to a point. This fact is restored by the computation of the quantity with a gauge fixing corresponding to either $U_{\mu\nu} = 1$ or -1. The vanishing result is due to the symmetric structure consisting of $U_{\mu\nu} = 1$ and -1 on the constraint surface.

$\langle S_{\Delta_3}\rangle$ is the expectation value of the action. It is invariant under SU(2) gauge transformations and spacetime diffeomorphisms. The value of the action on the constraint surface is identically zero. Therefore, $\langle S_{\Delta_3}\rangle$ must vanish if the path integral successfully eliminates the contributions from the degrees of freedom constrained. This must be the case if the path integral adopts the exponential oscillation form with $\beta \to \infty$. On the other hand, if the path integral adopts the exponential decay form, then the dominant contribution comes from the value of the action far away from zero and hence $\langle S_{\Delta_3}\rangle$ unlikely vanishes. This fact can be checked by replacing β in the path integral by $i\beta$.

In order to compute $\langle S_{\Delta_3}\rangle$, take a derivative of (31) with respect to β (with i multiplied) instead of inserting S_{Δ_3} into (29). This procedure results in the following quantity inserted in the path integral.

$$\sum_{x\in\Delta}\sum_{\substack{(\mu\nu)=(12),\\(20),(01)}}\frac{i\frac{d}{d\beta}\Delta_{\mu\nu}(\beta)}{\Delta_{\mu\nu}(\beta)} \tag{34}$$

Since there is no particular place or direction in purely empty spacetime, the every term must have equal contrubution and we compute one of the terms as follows.

$$\frac{i\frac{d}{d\beta}\Delta_{\mu\nu}(\beta)}{\Delta_{\mu\nu}(\beta)} = -\frac{4i}{\beta}+\frac{4i[J_{2l_{\mu\nu}+1}(2\beta)]^2}{\beta^3\Delta_{\mu\nu}} \to -\frac{4i}{\beta}\left(1-\frac{1}{c\pi}\cos^2[2\beta-\frac{\pi}{2}(2l_{\mu\nu}+1)-\frac{\pi}{4}]\right), \tag{35}$$

as β goes to infinity. Here, we have used the asymptotic formula $J_m(x) \to \sqrt{\frac{2}{\pi x}}\cos(x-\frac{m\pi}{2}-\frac{\pi}{4})$ for large x and the fact that $\Delta_{\mu\nu}$ goes to $\beta^{-4}\delta(0)$ or more precisely $c\beta^{-3}$ as $\beta \to \infty$, where c is a constant. We do not need the detailed value of c. This fact can be understood by rescaling β in (32). From (35), we conclude $\langle S_{\Delta_3}\rangle \to 0$ as $\beta \to \infty$. Note that if one replaces β by $i\beta$ in the path integral, then one finds (35) with the cosine replaced by the hyperbolic cosine. In this case, $\langle S_{\Delta_3}\rangle$ diverges instead of converging to zero as $\beta \to \infty$.

5 Conclusion

In the present work, we constructed a surface-theoretic lattice quantum gravity model in 4 dimensional Riemannian spacetime based on the SU(2) Ashtekar formulation of general relativity. We introduced a version of the action of general relativity and defined its lattice version on a fixed hyper cubic lattice. We introduced a dimensionless "(inverse) coupling" constant so that the magnitude of the action is finite per lattice point. We defined a path integral whose integrand has the exponential oscillation form so that it eliminates the degrees of freedom constrained. The finiteness of the action with finite lattice allowed the path integral without gauge fixing finite. We expanded the path integral in the SU(2) characters and showed that the path integral can be written as a sum over surface-like excitations in spacetime. We showed that

the 3 dimensional version of the model exists and its path integral is reduced to that of 3-dimensional lattice BF theory. Therefore, we considered the model as a 4 dimensional generalization of the Ponzano-Regge model with local degrees of freedom. We examined the expectation values of two basic variables of the theory, the Wilson loop and the action of the model in 3 dimensions and showed all of them vanish. We discussed the meaning of each of them and understood that the model has a chance of capturing the physical degrees of freedome of general relativity.

A Useful formulae

In the present work, we have used the following mathematical formulae.

$$e^{ix\frac{1}{2}\mathrm{Tr}U} = \sum_j 2\frac{2j+1}{x} i^{2j} J_{2j+1}(x)\chi_j(U), \tag{36}$$

$$J_m(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta e^{ix\sin\theta - im\theta}. \tag{37}$$

The first formula can be easily proved and the second is a definition of the Bessel function of the first kind. $\chi_j(U)$ is the character of SU(2) element U in the spin-j representation and j is a spin taking values 0, $\frac{1}{2}$, 1, $\frac{3}{2}\cdots$.

$$(-1)^{n-m} D^{(j)}_{nm}(U) = D^{(j)*}_{-n,-m}(U), \tag{38}$$

$$D^{(j_1)}_{n_1m_1}(U) D^{(j_2)}_{n_2m_2}(U) = \sum_{j,n,m}(2j+1)\begin{pmatrix} j_1 & j_2 & j \\ n_1 & n_2 & n \end{pmatrix}\begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{pmatrix} D^{(j)*}_{nm}(U), \tag{39}$$

$$\int dU D^{(i)}_{mn}(U) D^{(j)*}_{m'n'}(U) = \frac{1}{2j+1}\delta_{ij}\delta_{mm'}\delta_{nn'}, \tag{40}$$

$D^{(j)}_{mn}(U)$ is the spin-j representation matrix of SU(2) element U and m and n run from $-j$ through j with the increment 1. $\begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{pmatrix}$ is the so-called 3j-coefficient and defined by

$$\begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{pmatrix} := \frac{(-1)^{j_1-j_2-m}}{\sqrt{2j+1}}\langle j_1m_1; j_2m_2|j, -m\rangle, \tag{41}$$

with the Clebsch-Gordan coefficient $\langle j_1m_1; j_2m_2|jm\rangle$. The asterisks mean the complex conjugate and the sum of j is taken over $|j_1 - j_2|$ through $j_1 + j_2$ and the sums of n and m over $-j$ through j.

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